School Choice and Efficient Priority-Based Resource Allocation

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In many countries, children were automatically sent to a school in their neighborhoods.

Recently, more and more cities in the United States and in other countries employ school choice programs: school authorities take into account preferences of children and their parents.

Typical goals of school authorities are: (1) efficient placement, (2) fairness of outcomes, (3) easy for participants to understand and use, etc.
Model (Abdulkadiroglu and Sonmez 2003, AER)

Finite sets $S$ of students and $C$ of schools.

Each student can be matched to at most one school, and each school can admit at most $q_c$ students.

Each student $s$ has strict preferences $\succ_s$ over schools and being unmatched (denoted by $\emptyset$).

For each school, there is a (for now, strict) priority order over students. $s \succ_c s'$ means “student $s$ has higher priority for $c$ than $s'$.

The outcome is a matching (denoted $\mu$), which specifies which student attends which school. $\mu(s)$ is the school that student $s$ attends, and $\mu(c)$ is the set of students that $c$ admits.
A matching is **stable** if there is

1. **No blocking individual.** \( \mu(s) \) is acceptable to each student \( s \), each \( s \in \mu(c) \) is acceptable to \( c \) for each school \( c \), and \( |\mu(c)| \leq q_c \).
2. **No blocking pair.** There is no pair \( s \) and \( c \) such that
   1. \( c >_s \mu(s) \) and
   2. \( |\mu(c)| < q_c \) and \( s >_c \emptyset \), or \( s >_c s' \) for some \( s' \in \mu(c) \).
Stability as Fairness Criterion

In school choice, stability can be understood as a fairness criterion.

No blocking individual simply means no one is forced to attend an unacceptable school, and only qualified students can be admitted to a school (in some districts, all students are acceptable. Such cases are special cases.)

No blocking pair means no justified envy. That is, there is no situation in which student $s$ is matched to a worse school than school $c$, and $c$ admits another student who has lower priority at $c$ than $s$ does.

So stability may be a reasonable property we want for school choice mechanisms.

The stability concept is used for labor markets as well.
Boston mechanism

The **Boston mechanism**: a common mechanism.

- **Step 0**: Each student submits a preference ranking of the schools.
- **Step 1**: In Step 1 only the top choices of the students are considered. For each school, consider the students who have listed it as their top choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her top choice.
- **Step k**: Consider the remaining students. In Step k only the $k^{th}$ choices of these students are considered. For each school still with available seats, consider the students who have listed it as their $k^{th}$ choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as her $k^{th}$ choice.
Boston mechanism has lots of problematic features:

1. The result is not necessarily stable (fair).
2. It is not strategy-proof.
3. Moreover, it is easy to manipulate it. Even if a student has a very high priority at a school, unless she lists it as her top choice she loses her priority to students who have top ranked that school.
Worries in Boston mechanism is real

St. Petersburg Times (09/14/2003):

Make a realistic, informed selection on the school you list as your first choice. It’s the cleanest shot you will get at a school, but if you aim too high you might miss. Here’s why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That’s because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

The 2004-2005 BPS School Guide:

”For a better choice of your ‘first choice’ school . . . consider choosing less popular schools.”
Stable matchings always exist in school choice problems

**Theorem (Gale and Shapley 1962)**

There exists a stable matching in any school choice problem.

**Proof: Gale and Shapley (1962).**

The theorem gives some hope that a fair assignment may be possible in school choice.
Student-optimal stable matchings

Theorem (Gale and Shapley 1962; RS Theorem 2.12)

There exists a student-optimal stable matching, that is, a stable matching that every student weakly prefers to any stable matching. The result of the student-proposing DA algorithm is the student-optimal stable matching.

So the welfare is maximized by the student-proposing DA, subject to stability.
The student-proposing DA is strategy-proof. That is, telling the truth is a dominant strategy for every student.

In fact, DA is the only strategy-proof stable mechanism.
Other good properties of DA

Other good properties are known for DA.

1. DA is the only stable mechanism that respects improvements, that is, a higher priority is always good for a student (Balinski and Sonmez 1999).

2. DA is weakly Pareto optimal, that is, there is no other individually rational matching that every student strictly prefers to DA (Roth 1982; extended by Kojima 2008 and Hatfield and Kojima forthcoming).

So the student-proposing DA is the big winner among all stable mechanisms.

Balinski and Sonmez (1999) show that the mechanism used for college admission in Turkey is equivalent to the school-proposing DA, and advocated the change of the mechanism to the student-proposing DA.

Boston and New York City changed their adopted DA for their school choice systems, and they are in use now.
Efficiency cost of stability

Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

$$\succ_i : b, a$$
$$\succ_j : a,$$
$$\succ_k : a, b,$$

and both schools have one position and priorities are

$$\succ_a : i, j, k,$$
$$\succ_b : k, i.$$  

The result of DA is

$$\mu = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix}$$

This is less preferred by every student to

$$\mu' = \begin{pmatrix} i & j & k \\ b & \emptyset & a \end{pmatrix}$$

so DA is not Pareto efficient.
Based on Ergin (2002).

When is DA costly in school choice? What priority structure ensures efficiency of DA?

To investigate this issue, it is useful to see (now familiar) example where DA is inefficient.

Assume that every student is acceptable to every school throughout.
Efficiency cost of stability

Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

\[ i \succ b, a \]
\[ j \succ a, \]
\[ k \succ a, b, \]

and both schools have one position and priorities are

\[ a \succ i, j, k, \]
\[ b \succ k, i. \]

The result of DA is

\[ \mu = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix} \]

This is less preferred by every student to

\[ \mu' = \begin{pmatrix} i & j & k \\ b & \emptyset & a \end{pmatrix} \]

so DA is not Pareto efficient.
Why inefficiency happens in DA

There is a chain of applications and rejections in the above example.

Such a chain is caused by a “cycle” of priorities, that is, two schools’ priorities are

\[ \succ_a : i, j, k \]
\[ \succ_b : k, i. \]

Because of such a cycle, in DA,

1. \( k \) applies to her favorite \( a \) but \( j \) displaces \( k \),
2. \( k \) is forced to apply to her second choice \( b \), displacing \( i \) from his favorite \( b \),
3. \( i \) is forced to apply to his second choice \( a \), displacing \( j \).

In the end, \( j \) is displaced by school \( a \) anyway, with the result being just causing more rejections and making \( i \) and \( k \) worse off.

Ergin says the priority structure of the schools is **acyclic** if, roughly speaking, there is no such cycle.
Definition of Acyclicity

We need to be a bit more careful for the formal definition of DA: Specifically, if the good is not scarce, then the chain of rejections won’t occur. Let $U_a(i) = \{j \in S | j \succ_a i\}$.

**Definition**

A **cycle** is a set $a, b \in C$ and $i, j, k \in S$ such that

1. $i \succ_a j \succ_a k, k \succ_b i$,
2. There exist (possibly empty) disjoint sets of students $S_a, S_b \in S \setminus \{i, j, k\}$ such that $S_a \subseteq U_a(j), |S_a| = q_a - 1$, $S_b \subseteq U_b(i), |S_b| = q_b - 1$.

A profile of school priorities and capacities is **acyclic** if there exists no cycle.
Acyclicity and Pareto efficiency

Theorem (Ergin 2002)

*DA is Pareto efficient for all possible student preferences if and only if the priority structure of the schools is acyclic.*

This theorem seems to be bad news for school choice, because most priority structures are cyclic (acyclicity allows for very little heterogeneity in students’ rankings across schools; see Theorem 2 of Ergin).
The “only if” part (i.e., efficient $\rightarrow$ acyclic) is relatively easy (once we notice the trick): Construct a counterexample like previous ones.

The “if” part (acyclic $\rightarrow$ efficient) is surprising: The definition seems to eliminate only a very special case of inefficiency (like the above example). So, why is it also sufficient?
Proof: Basic Issues

We will see the proof of “acyclic $\rightarrow$ efficient” for the simple case in which $q_c = 1$ for all $c \in C$ (the proof for the general case is analogous but more complicated, see also a comment by Narita, 2009).

Definition (for the case in which $q_c = 1$ for every $c \in C$)

A **generalized cycle** is a set $a_0, a_1, \ldots, a_{n-1} \in C$ and $j, i_0, i_1, \ldots, i_{n-1} \in S$ with $n \geq 2$ such that

$$i_0 \succ a_0 \, j \succ a_0 \, i_{n-1} \succ a_{n-1} \, i_{n-2}, \ldots, i_1 \succ a_1 \, i_0.$$ 

The definition coincides with the one for a cycle if $n = 2$.

Proof strategy: Show that (1) “inefficiency $\rightarrow$ a generalized cycle exists”, and then (2) “a generalized cycle $\rightarrow$ a cycle (i.e., a generalized cycle with $n = 2$).”
Proof Step 1: “inefficiency $\rightarrow$ a generalized cycle exists”

Assume the outcome of DA $\mu$ is not Pareto efficient. Then there is another matching $\mu'$ that Pareto dominates it.

Let $S' = \{s \in S | \mu'(s) \succ_s \mu(s)\}$. Note that $S' \neq \emptyset$.

Each $s \in S'$ is rejected by $\mu'(s)$ at some step of DA (why?).

$s' := \mu^{-1}(\mu'(s)) \in S'$ if $s \in S'$ (What does this mean? Why is this true?)

So, form a set of schools and students $a_0, a_1, \ldots, a_{n-1} \in C$ and $i_0, i_1, \ldots, i_{n-1} \in S$ with $n \geq 2$ such that

$$i_0 \succ_{a_0} i_{n-1}, i_{n-1} \succ_{a_{n-1}} i_{n-2}, \ldots, i_1 \succ_{a_1} i_0.$$

This set almost forms a generalized cycle, but not quite (where is the difference?).
In Search of $j$

We have not found $j$ in the definition of a generalized cycle yet, so find her.

Let $r$ be the last step in which someone in $\{i_0, \ldots, i_{n-1}\}$ applies to (and is accepted by) a school: Without loss of generality, call her $i_0$.

At the end of step $r - 1$, $a_0$ should be matched with someone $j \neq i_0, \ldots, i_{n-1}$ (why?).

Then we should have $i_0 \succ_{a_0} j \succ_{a_0} i_{n-1}$, thus finding a generalized cycle.
Proof Step 2: “a generalized cycle $\rightarrow$ a cycle (i.e., $n = 2$).”

Suppose there is a generalized cycle with length $n > 2$. Consider two cases, (1) $i_0 \succ_a i_2$, and (2) $i_2 \succ_a i_0$.

1. Suppose $i_0 \succ_a i_2$. Combine this with the maintained assumption $i_2 \succ_a i_1$ and $i_1 \succ_a i_0$. Then $i_0 \succ_a i_2 \succ_a i_1$ and $i_1 \succ_a i_0$, meeting the definition of a cycle with $(i, j, k, a, b) = (i_0, i_2, i_1, a_2, a_1)$.

2. Suppose $i_2 \succ_a i_0$. Then, by the maintained assumption that $i_0 \succ_a i - 1 \succ_a \cdots \succ_a i_1 \succ_a i_2$, we obtain $i_0 \succ_a i - 1 \succ_a \cdots \succ_a i_3 \succ_a i_2, i_2 \succ_a i_0$, showing that there is a generalized cycle of length $n - 1$.

This completes the proof.
Group Strategy-proofness and DA

A mechanism is **group strategy-proof** if there is no subset $S'$ of students and their jointly misreported preferences such that everyone in $S'$ is made weakly better off and at least one strictly better off in the assignment under the misreported preferences than under truth-telling.

Is DA group strategy-proof?

Aside: there is a weaker notion of group strategy-proofness, requiring that no subset of students can jointly misreport preferences and make every member strictly better off. This version of group strategy-proofness is satisfied by DA (Dubins and Freedman 1981; Hatfield and Kojima 2008, 2009)
DA is not necessarily group strategy-proof

Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

\[
\succ_i : b, a \\
\succ_j : a, \\
\succ_k : a, b,
\]

and both schools have one position and priorities are

\[
\succ_a : i, j, k, \\
\succ_b : k, i.
\]

The result of DA is

\[
\mu = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix}
\]

Consider a coalition $S' = S$ where $j$ reports $\emptyset$ as her first choice while $i$ and $k$ report their true preferences. Then the resulting matching,

\[
\mu' = \begin{pmatrix} i & j & k \\ b & \emptyset & a \end{pmatrix}
\]

is strictly preferred by $i$ and $k$ and weakly by $j$ to $\mu$. 
Acyclicity and group strategy-proofness

Theorem (Ergin 2002)

*DA is group strategy-proof if and only if the priority structure of the schools is acyclic.*

In particular, Pareto efficiency and group strategy-proofness are equivalent among DA mechanisms (this seems like an unexpected result!)

An (incomplete) intuition: a cyclic structure means a cycle: Those involved in a cycle can jointly “withdraw” from applications for some schools to eliminate a chain reaction of rejections.
Extensions

Ehlers and Erdil (2009) extend the condition to non-strict priorities (as is common in school choice)

Kumano (2009) extends the condition to non-responsive priorities, namely the class of “substitutable and acceptant” priority structures as defined by Kojima and Manea (2009).

Cases with “matching with contracts” (Pakzad-Hurson, 2013): A general theory.

Additional benefit of acyclicity: the same condition turns out to be crucial in other applications (discussed later). So the condition seems to be of interest beyond the specific application here.
How restrictive are acyclic priorities?

The acyclicity seems pretty restrictive, but how restrictive, exactly?

Ergin gives a characterization in his Theorem 2. For the case where $q_c = 1$ for all $c \in C$, the theorem says:

**Theorem**

*The priority structure is acyclic if and only if the ranking of each student differs at most by one across schools.*
Open questions

The acyclicity conditions is a very stringent one, and seems unlikely to be satisfied in most applications.

There are some open questions (including how to formalize the issues), such as

1. Is there any way to quantify the efficiency of a given priority structure, like “if the structure is ‘almost acyclic’, then the efficiency loss is small.”
2. Can we say that one priority structure is “more efficient” than another?
3. What is the class of mechanisms under which Pareto efficiency and group strategy-proofness is equivalent?

Summary

In school choice (or any resource allocation problem with priority), DA can be easily adapted to produce a fair and constrained efficient matching.

We saw a characterization, in terms of the priority structures, of the necessary and sufficient conditions for DA to be efficient.

The condition turns out to be very restrictive: allows only small variations in student rankings across schools.

Interestingly, the condition turns out to appear in applications that are very different at a first glance.
Based on Kojima (2010, TE).

In resource allocation, DA is both stable (fair) and strategy-proof. This makes it regarded as a good mechanism.

But what about a combined manipulation? That is, first misreport preferences and then file for a re-matching (blocking)?

The setup is intended to model appeals processes (in NYC, about 5,000 students out of 90,000 file for appeals under DA; 300 among them are from those who were matched to their first choices).
DA is not immune to combined manipulation

Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

\[
\succ_i : b, a \quad \succ_j : a, \quad \succ_k : a, b,
\]

and both schools have one position and priorities are

\[
\succ_a : i, j, k, \quad \succ_b : k, i.
\]

The result of DA is

\[
\mu = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix}
\]

Suppose $j$ reports that $\emptyset$ is the first choice. Then the resulting matching is

\[
\mu' = \begin{pmatrix} i & j & k \\ b & \emptyset & a \end{pmatrix}
\]

Because $j \succ_a k$, $j$ could ask to be admitted to $a$; if granted, $j$ is made better off.
Robust stability

A mechanism is said to be **robustly stable** (Chakraborty et al. 2009; adapted by Kojima 2010) if

1. it is a stable mechanism,
2. it is strategy-proof, and
3. it is immune to any combination of misreporting and ex-post rematching (blocking).

The last example shows that DA is not generally robustly stable.

The example actually shows that there does not exist any robustly stable mechanism (why?).
Priority structure and robust stability

Robust stability is not achievable in general.

A question: under what condition on the priority structure does there exist a robustly stable mechanism?
Acyclicity and Robust Stability

Definition (Ergin 2002)

A **cycle** is a set \( a, b \in C \) and \( i, j, k \in S \) such that

1. \( i \succ_a j \succ_a k, \ k \succ_b i \),
2. There exist (possibly empty) disjoint sets of students \( S_a, S_b \in S \setminus \{i, j, k\} \) such that \( S_a \subseteq U_a(j), \ |S_a| = q_a - 1, S_b \subseteq U_b(i), \ |S_b| = q_b - 1 \).

A profile of school priorities and capacities is **acyclic** if there exists no cycle.

Theorem

*Given the priority structure, the following statements are equivalent.*

1. There exists a robustly stable mechanism.
2. DA with the given priority structure is robustly stable.
3. The priority structure is acyclic.

This theorem seems to be bad news for school systems: most priority structures violate acyclicity.
Proof: Issues

Since DA is the only strategy-proof stable mechanism, “there exists a robustly stable mechanism” and “DA is robustly stable” are obviously equivalent.

So consider robust stability of DA.
The claim “DA is robustly stable $\rightarrow$ priority structure is acyclic” is relatively easy: Construct a counterexample like previous ones.

Example: Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

\[ i \succ b, a \]
\[ j \succ a \]
\[ k \succ a, b \]

and both schools have one position and priorities are

\[ a \succ i, j, k \]
\[ b \succ k, i \]
Proof: Preliminary

The other direction ("priority structure is acyclic $\rightarrow$ DA is robustly stable") is more involved: why is it the case?

A mechanism is said to be nonbossy (Satterthwaite and Sonnenschein, RES 1981) if: “if an agent changes her reported preferences but her matching does not change, then matching for no one else changes.”

Proposition (Ergin 2002)

*DA is nonbossy if and only if the priority structure is acyclic.*

For intuition, see the familiar example.

Aside: Actually, this is what we have already seen: DA is group strategy-proof if and only if the priority structure is acyclic (Ergin, Theorem 1). Lemma 1 of Papai (EMA 2000) shows that GSP is equivalent to strategy-proofness and nonbossiness.

We use Ergin’s result in the proof.
Proof for Case 1: \( \varphi^S_s(\succ') = \emptyset \)

Denote by \( \varphi \) the DA mechanism. For contradiction, suppose that the priority structure is acyclic but DA is not robustly stable. Since DA is stable and strategy-proof, this assumption implies that

**Condition (A):** there exist \( s \in S \), \( c \in C \), student preference profile \( \succ := (\succ_s)_{s \in S} \) and \( \succ'_s \) such that

1. \( c \succ_s \varphi^S_s(\succ) \), and
2. \( s \succ_c s' \) for some \( s' \in \varphi^S_c(\succ'_s, \succ_{-s}) \) or \( |\varphi^S_c(\succ'_s, \succ_{-s})| < q_c \).

Letting \( \succ' = (\succ'_s, \succ_{-s}) \), we consider the following cases.

(Case 1) Suppose \( \varphi^S_s(\succ') = \emptyset \). Let \( \succ'' : c, \emptyset \) and \( \succ'' = (\succ''_s, \succ_{-s}) \).

1. Suppose \( \varphi^S_s(\succ'') = \emptyset \). Then, since the priority structure is acyclic by assumption and hence \( \varphi^S \) is nonbossy (Ergin’s result), we have \( \varphi^S(\succ'') = \varphi^S(\succ') \). So,
   1. \( c \succ''_s \emptyset = \varphi^S_s(\succ'') \), and,
   2. by **Condition (A)**, either \( s \succ_c s' \) for some \( s' \in \varphi^S_c(\succ') = \varphi^S_c(\succ'') \) or \( |\varphi^S_c(\succ'')| = |\varphi^S_c(\succ')| < q_c \)

So \( \varphi^S(\succ'') \) is unstable under \( \succ'' \). This contradicts the fact that \( \varphi^S \) is a stable mechanism.

2. Suppose \( \varphi^S_s(\succ'') = c \). Then this is a contradiction to strategy-proofness of \( \varphi^S \), since \( \varphi^S_s(\succ'') = c \succ_s \varphi^S_s(\succ) \).
Proof for Case 2: $\varphi^S_s(\succ') \neq \emptyset$

(Case 2) Suppose $\varphi^S_s(\succ') \neq \emptyset$. Let $\succ''_s: \emptyset$, and $\succ'' = (\succ''_s, \succ_{-s})$.

By the well-known comparative statics (Kelso and Crawford 1982, Gale and Sotomayor 1985), the worst student in $\varphi^S_c(\succ'')$ has (weakly) lower priority at $c$ than the worst student in $\varphi^S_c(\succ')$.

Thus

1. Condition (A) is satisfied with respect to $s, c$ and $\succ''_s$ (instead of $\succ'_s$) and,
2. $\varphi^S_s(\succ'') = \emptyset$,

so the analysis reduces to Case (1).
Combined manipulations by groups

Afacan (2011, GEB) complemented the result by Kojima (2010) by considering combined manipulations by groups of students.

As in the case with Pareto efficiency and group strategy-proofness, there could be (at least) two definitions of group robust stability, requiring that there is no group manipulation causing

1. strict improvement for everyone in the manipulating coalition, or
2. weak improvement for everyone, with at least one strict

For the first concept (weaker requirement), it turns out that Ergin’s acyclicity is also a necessary and sufficient condition for non-manipulability.

For the second concept (stronger requirement), the mechanism may be manipulable even with acyclic priority structures.
Constrained School Choice

Based on Haeringer and Klijn (JET 2009).

DA is regarded as a desirable mechanism partly because it is strategy-proof.

But in many school districts, students are required to submit a short (constrained) list of schools.
Short list in the field

NYC school choice: 12 schools

Boston school choice: 5 schools (until 2006)

Spain college admission: 8 colleges

Hungary college admission: 4 colleges

Tokyo school choice: 1 school

List length seems to be binding to some extent: roughly 25 percent of students list the maximum 12 schools in NYC.
A constraint on the list length means that the mechanisms cannot be strategy-proof: “truth telling” is simply infeasible!

A question: if students strategically submit their preferences in constrained mechanisms, what are the equilibrium outcomes? Do they coincide with stable outcomes?

Haeringer and Klijn analyze Nash equilibrium outcomes to study these issues.
Every stable matching can be a Nash equilibrium outcome

Proposition

For any stable matching, there is a Nash equilibrium of DA (with any list length constraint) resulting in that stable matching.

Proof: Fix a stable matching.

1. Consider a strategy profile where each student lists the school matched in that stable matching as the only acceptable school.
2. Clearly, the outcome is the prescribed stable matching.
3. It is also easy to verify that this is a Nash equilibrium.

Is there any other matching supported by a Nash equilibrium?
Unstable matching may be a Nash equilibrium outcome

Let \( S = \{i, j, k\} \), \( C = \{a, b\} \), and student preferences are

\[
\succ_i : b, a \\
\succ_j : a, \\
\succ_k : a, b,
\]

and both schools have one position and priorities are

\[
\succ_a : i, j, k, \\
\succ_b : k, i.
\]

Consider a strategy profile where \( j \) declares \( \emptyset \) to be her first choice, whereas \( i \) and \( k \) report their preference truthfully.

It is a Nash equilibrium, and the resulting matching

\[
\mu = \begin{pmatrix} i & j & k \\ b & \emptyset & a \end{pmatrix}
\]

is an unstable matching.

Question: Any condition on the priority structure where Nash equilibrium outcomes coincide with the set of stable matchings?
From the example, it seems that acyclicity is key (or at least necessary) for Nash equilibrium outcomes to coincide with stable matchings.

In fact, Haeringer and Klijn show that

**Theorem (Haeringer and Klijn 2009)**

*For any constraint on the list length $\geq 2$, the set of Nash equilibrium outcomes under DA coincides with the set of stable matchings if and only if the priority structure of the schools is Ergin-acyclic.*

Remark: If the list length constraint is 1, then NE outcomes $=$ stable matchings in general.
We prove the claim for $q_c = 1$ for all $c \in C$ (see the paper for the general case).

The “only if” part (i.e., “NE=stable” $\rightarrow$ “Ergin-acyclic”) is relatively easy (as in Ergin 2002): Construct a counterexample like the previous ones.

The “if” part (i.e., “Ergin-acyclic” $\rightarrow$ “NE=stable”) is surprising and more involved.
For contradiction, assume there is a (true) preference profile $\succ$ and a Nash equilibrium $\succ'$ resulting in an unstable outcome.

Then there are students $i$ and $j$, and a school $c$ such that

$$c \succ_i \varphi_i(\succ'), c = \varphi_j(\succ'), i \succ_c j.$$  \hfill (1)

By strategy-proofness of DA (if no list-length constraint),

$$\varphi_i(\succ_i, \succ'_{-i}) \succeq_i \varphi_i(\succ').$$  \hfill (2)

Let $\succ''_i$ be a preference regarding only $\varphi_i(\succ_i, \succ'_{-i})$ as acceptable (note that this is a feasible strategy for any list-length constraint!).

One can see that (Lemma A.1 in Haeringer and Klijn; why?)

$$\varphi_i(\succ''_i, \succ'_{-i}) = \varphi_i(\succ_i, \succ'_{-i})$$  \hfill (3)

By (??) and (??), we have $\varphi_i(\succ''_i, \succ'_{-i}) \succeq_i \varphi_i(\succ'_i, \succ'_{-i}).$

But $\succ'$ is a Nash equilibrium, so $\varphi_i(\succ''_i, \succ'_{-i}) = \varphi_i(\succ'_i, \succ'_{-i}).$

By (??), this implies $\varphi_i(\succ_i, \succ'_{-i}) = \varphi_i(\succ'_i, \succ'_{-i}).$

DA with acyclic priorities is non-bossy, so $\varphi(\succ_i, \succ'_{-i}) = \varphi(\succ').$

This and (??) mean $\varphi(\succ_i, \succ'_{-i})$ is unstable at $(\succ_i, \succ'_{-i})$, QED.
The Result about TTC

What about TTC?

**Theorem (Haeringer and Klijn 2009)**

For any constraint on the list length $\geq 2$, the set of Nash equilibrium outcomes under TTC coincides with the set of stable matchings if and only if the priority structure of the schools is Kesten-acyclic (Kesten 2006 JET).

Suppose schools (school principals?) have their own preference such that “I want to admit students according to a linear order up to capacity $q$ (called “responsive”).”

Assume priority $\succ c$ is given by law, but capacity $q$ should be asked. → Schools may have incentives to underreport capacities.

Deputy Chancellor of Schools described principals concealing capacity as a major issue with their previous unstable mechanism (New York Times (11/19/04)):

“Before you might have a situation where a school was going to take 100 new children for 9th grade, they might have declared only 40 seats, and then placed the other 60 outside of the process.”
DA can be manipulated by underreporting capacities

Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

$\succ_i : b, a$

$\succ_j : a,$

$\succ_k : a, b,$

and schools’ true capacities are $q_a = 2$, $q_b = 1$, and priorities are

$\succ_a : i, j, k,$

$\succ_b : k, i.$

DA results in

$$\mu = \begin{pmatrix} i & j & k \\ b & a & a \end{pmatrix}$$

Suppose that school $a$ likes 1 better than $\{2, 3\}$. Suppose that $a$ reports capacity $q'_a = 1$. Then DA results in

$$\mu' = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix},$$

a better outcome for $a$. 
Impossibility Theorem

DA is manipulable by capacity manipulations.

Sonmez (JET 1997) further shows that there is no stable mechanism immune to capacity manipulations.

Question: Under what conditions is DA immune to capacity manipulations?
Priority Structure and Capacity Manipulations

Note that, in the above example, the original priority structure does not violate acyclicity: School seats are not scarce enough.

On the other hand, the priority structure after school $a$ underreports its capacity does violate acyclicity.

For each school, Kesten introduces the “minimum capacity” the school district allows the school to report. Denote the minimum capacity vector by $q$.

**Theorem**

$DA$ is immune to capacity manipulation for all school preferences if and only if the priority structure $(\succ, q)$ is Ergin-acyclic.
The intuition that acyclicity is necessary seems somewhat different from other papers, but shares a basic insight: Capacity manipulation can benefit a school (only) because doing so can cause a rejection chain that returns to it.

The chain can return only if there is heterogeneity in priorities across schools (this gives partial intuition).

But why a cycle as defined by Ergin, not merely some heterogeneity?
Let $S = \{i, j, k\}$, $C = \{a, b\}$, and student preferences are

$\succ_i : b, a$

$\succ_j : a,$

$\succ_k : a, b,$

and schools’ true capacities are $q_a = 2$, $q_b = 1$, and priorities are

$\succ_a : i, j, k,$

$\succ_b : k, i.$

DA results in

$\mu = \begin{pmatrix} i & j & k \\ b & a & a \end{pmatrix}$

Suppose that school $a$ likes 1 better than $\{2, 3\}$. Suppose that $a$ reports capacity $q_a' = 1$. Then DA results in

$\mu' = \begin{pmatrix} i & j & k \\ a & \emptyset & b \end{pmatrix},$

a better outcome for $a$. 
Multi-unit Demand

Based on Kojima (2011).

Ergin considered efficient and group-strategy-proof assignment mechanism under single-unit demand.

What if each student may demand more than one good? (Course allocation or sports draft, for instance)

Casual observation: Under multiunit demand, the DA algorithm is not (to my knowledge) used in practice (see Sonmez and Unver IER 2010).
Efficiency under multiunit demand

Let $S = \{i, j\}$, $C = \{a, b\}$. Student $i$ wants two courses and student $j$ wants one course. Preferences are

- $\succ_i : b, a$
- $\succ_j : a, b$

and both schools have one position and priorities are

- $\succ_a : i, j,$
- $\succ_b : j, i.$

The result of DA is

$$\mu = \begin{pmatrix} i & j \\ a & b \end{pmatrix}$$

This is less preferred by every student to

$$\mu' = \begin{pmatrix} i & j \\ b & a \end{pmatrix}$$

so DA is not Pareto efficient... actually it is not even weakly Pareto efficient.
Let $S = \{i, j\}, C = \{a, b\}$. Student $i$ wants two courses and student $j$ wants one course. Preferences are

$\succ_i : b, a$

$\succ_j : a, b$

and both schools have one position and priorities are

$\succ_a : i, j$

$\succ_b : j, i$

The result of DA is

$\mu = \begin{pmatrix} i & j \\ a & b \end{pmatrix}$

If $i$ misreports that only $b$ is acceptable, then DA results in

$\mu' = \begin{pmatrix} i & j \\ b & a \end{pmatrix}$

so DA is not even strategy-proof.
Characterization

Under multi-unit demand, even weak Pareto efficiency or strategy-proofness are violated.

Even worse, the priority structure

\[ \succ_a : i, j, \]
\[ \succ_b : j, i. \]

was acyclic in the last example, but it didn’t help.

We say that the priority structure is **essentially homogeneous** if (if \( q_c = 1 \) for all \( c \)), the ranking of all courses are exactly the same.

**Theorem**

The following conditions are equivalent.

1. DA is Pareto efficient.
2. DA is strategy-proof.
3. The priority structure is essentially homogeneous.
School choice with consent

The idea that inefficiency of DA is caused by a “rejection chain” seems to be of a broader implication.

Kesten (QJE forthcoming) proposes a mechanism that improves upon DA in terms of efficiency.

Acyclicity is not explicit, but the basic idea seems similar: eliminate a chain reaction of costly applications and rejections by “editing preferences” submitted by students.
Characterization of DA

A chain reaction of costly applications and rejections, caused by Ergin-cycles, stands out as a feature of DA.

Kojima and Manea (EMA forthcoming) find that this is (almost) characterize DA, indeed.

They define a new axiom “weak Maskin monotonicity,” a condition saying that applications to “no chance schools” hurt other students, and show that it characterizes DA with a few more standard axioms.
Summary

We saw a few applications of the acyclicity conditions.

Although it was introduced in a specific context (Pareto efficiency of DA), the same condition turns out to be crucial in different contexts (robust stability, constrained school choice, capacity manipulation).

Although these applications look pretty different from the original question, the intuition seems to have much in common (remember, the same example gave the intuition for many different issues!).

This is probably why the same condition appears over and over again.

The condition seems to be closely related to the mechanics of DA, so there may be more applications.