No stable matching mechanism is obviously strategy-proof

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Abstract

The (men-proposing) deferred acceptance algorithm, which finds the men-optimal stable matching in a two-sided marriage market, is known to be strategy-proof for all men. This algorithm is now used extensively by many centralized matching markets. Although applicants are being advised that it is in their best interest to state their true preferences, empirical evidence suggests that a significant fraction are nonetheless attempting to strategically misreport their preferences. We show that while strategy-proof for all men, no mechanism implementing the men-optimal stable matching is obviously strategy-proof for all men, a term recently defined by Li (2015). Consequently, no stable matching mechanism is obviously strategy-proof for all men. This implies that in any implementation of the men-optimal stable matching, significant cognitive effort and contingent reasoning are needed to be convinced that no strategic opportunities exist for any man.

1 Introduction

The study of two-sided matching markets and stability, initiated in the seminal paper of Gale and Shapley (1962), has grown to become a flourishing field, encompassing both theoretical and experimental work. In a matching market comprised of women and men, a (one-to-one) matching is stable if no woman and man prefer each other over their matches. Stability has proved to be useful not only to predict outcomes, but also as an important criteria in

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the success of designing centralized clearing houses for two-sided matching markets (Roth, 2002). In their paper, Gale and Shapley (1962) show that a stable matching exists in every matching market, and introduce the (men-proposing) deferred acceptance algorithm, which finds a stable matching called the men-optimal stable matching.

A highly desired property in the theory and practice of mechanism and market design is strategy-proofness, i.e., players having no incentives to misrepresent their preferences. As Dubins and Freedman (1981) show, the deferred acceptance algorithm is in fact strategy-proof for the proposing side, i.e., for all men. The guarantee of both stability and strategy-proofness has led many school choice districts and clearing houses for labor markets in the USA and across the world to adopt the deferred acceptance algorithm to find stable matchings in their respective two-sided markets. Interestingly, while advised that it is in their best interest to state their true preferences, empirical evidence suggests that a significant fraction of participants attempt (to no avail) to behave strategically and misreport their true preferences (Hassidim et al., 2015; Rees-Jones, 2015).

In a recent paper, Li (2015) formulates the idea that it is “easier to be convinced” of the strategy-proofness of some mechanisms over that of others. He introduces, and characterizes, the class of obviously strategy-proof mechanisms. Li shows that, roughly speaking, obviously strategy-proof mechanisms are those whose strategy-proofness can be proved even under a cognitively limited proof model that does not allow for contingent reasoning. For instance, this notion separates sealed-bid second-price auctions from ascending auctions (in which the bidder only needs to decide at any given moment whether or not to quit). In addition to analyzing auctions, Li (2015) analyzes a number of popular strategy-proof assignment/matching rules, such as Random Serial Dictatorship and Top Trading Cycles, and determines whether or not each of them is implementable via an obviously strategy-proof mechanism. In this paper, we ask whether it is possible to implement the men-optimal stable matching (or any other stable matching rule) via an obviously strategy-proof mechanism.

As Li (2015) illustrates, Random Serial Dictatorship (RSD), when implemented in a sequential manner, by letting each agent choose an object at her turn, is obviously strategy-proof. Similarly, when women’s preferences are perfectly aligned, the sequential “serial
dictatorship” implementation of the unique stable matching, by letting men choose in their ranked order, is also obviously strategy-proof. Generalizing to allow for weaker forms of alignment of women’s preferences, we show that if women’s preferences are acyclical (Ergin, 2002), the men-optimal stable matching can be implemented via an obviously strategy-proof mechanism. (While the obvious truthfulness of the basic questions we use to construct this implementation, namely, “do you prefer $x$ the most from of all currently unmatched women?”, draws from the same intuition as that of the RSD mechanism, the questions are considerably more flexible, and the order of the questions is more subtle.)

Our main result is that for general preferences, no mechanism that implements the men-optimal stable matching (or any other stable matching) is obviously strategy-proof for men. We first prove this impossibility result in a specifically crafted matching market with 3 women and 3 men, in which women have (cyclical) preferences that are a priori fixed and announced (and the men’s preferences are unrestricted). This immediately implies that any larger market, in which some 3 women have this structure of preferences over some 3 men, does not allow for an obviously strategy-proof implementation of the men-optimal stable matching. In particular, as the number of women or men grows large, if women’s preferences are fixed to preferences drawn independently and uniformly at random, with high probability no mechanism implementing the men-optimal stable matching is obviously strategy-proof for all men. In a two-sided scenario (i.e., where the women’s preferences are not fixed/announced in advance), this implies that even if some 3 women may have this structure of preferences over some 3 men, then no such implementation exists. In particular, if women’s preferences are unconstrained, then no mechanism that implements the men-optimal stable matching (or any other stable matching rule) is obviously strategy-proof for more than two men.

Combining our positive and negative results, no mechanism that implements the men-optimal stable matching is obviously strategy-proof for men as long as women’s preferences are “sufficiently unaligned”.

This paper sheds more light on fundamental differences between two-sided markets, where the mechanism objective is a two-sided notion such as stability, and other closely related markets, where the mechanism objective is a one-sided notion such as some variant of efficiency for one side or welfare maximization for one side. First, as noted, in assignment markets there exists an obviously strategy-proof ex-post efficient mechanism (RSD). Second, a variety of ascending auctions, from familiar multi-item auctions (Demange et al., 1986) to recently proposed clock auctions (Milgrom and Segal, 2014), maximize welfare or revenue and are obviously strategy-proof, despite the latter being based on deferred acceptance principles. In contrast, no stable mechanism in two-sided markets is obviously strategy-proof for any of the sides.

This paper is organized as follows. Section 2 provides the model and background, including the definition of obvious strategy-proofness in matching markets. Section 3 presents special cases for which an obviously strategy-proof implementation of the men-optimal stable matching exists. Section 4 provides the main impossibility result. Section 5 presents

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*A preference profile for women over men is cyclical if there are three men $a, b, c$ and two women $x, y$ such that $a \succeq_x b \succeq_x c \succeq_y a$. 

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corollaries in a model where the women’s preference are not fixed in advance. We conclude in Section 6.

2 Model and background

2.1 Matching with single-sided preferences

Let $M$ and $W$ be disjoint finite sets. In Sections 2.1 and 2.2, we think of $M$ as a set of agents and of $W$ as a set of resources.

Definition 1 (Preferences).

1. A preference list over $W$ is a totally-ordered subset of $W$ (if $w \in W$ does not appear on the preference list, then we think of $w$ as being considered unacceptable). Denote the set of all preference lists over $W$ by $\mathcal{P}(W)$.

2. A preference profile for $M$ over $W$ is a specification of a preference list over $W$ for each agent $m \in M$. (So the set of all preference profiles for $M$ over $W$ is $\mathcal{P}(W)^M$.)

3. Given a preference profile for $M$ over $W$, an agent $m \in M$ is said to prefer $w \in W$ over $w' \in W$, denoted by $w \succ_m w'$, if $w$ precedes $w'$ on $m$’s preference list or if $w$ is on $m$’s preference list while $w'$ is not. Moreover, $m$ is said to prefer every $w \in W$ that appears on his preference list over $m$ (which denotes $m$ being unmatched), and to prefer $m$ over every $w \in W$ not on his preference list. We write $w \succeq_m w'$ if it is not the case that $w' \succ_m w$.

Definition 2 (Matching). A matching between $M$ and $W$ is a one-to-one mapping between a subset of $M$ and a subset of $W$. Denote the set of all matchings between $M$ and $W$ by $\mathcal{M}(M,W)$. Given a matching $\mu$ between $M$ and $W$, for an agent $m \in M$ we write $\mu_m$ to denote $m$’s match (according to $\mu$), or write $\mu_m = m$ if $m$ is unmatched.

Definition 3 (Matching rule). A (single-sided) matching rule is a function $C : \mathcal{P}(W)^M \to \mathcal{M}(M,W)$, from preference profiles for $M$ over $W$ to matchings between $M$ and $W$.

Definition 4 (Strategy-proofness). Let $C$ be a matching rule.

1. $C$ is said to be strategy-proof for an agent $m \in M$ if for every preference profile $\bar{p} = (p_m)_{m \in M} \in \mathcal{P}(W)^M$ and for every alternative preference list $p'_m \in \mathcal{P}(W)$, it is the case that $C_m(\bar{p}) \succeq_m C_m(p'_m, \bar{p} - m)$ according to $p_m$, where $(p'_m, \bar{p} - m)$ denotes the preference profile obtained from $\bar{p}$ by setting the preferences of $m$ to be $p'_m$. In other words, $m$ would not be better off misrepresenting his preference list to be $p'_m$ instead of $p_m$.

2. $C$ is said to be strategy-proof if it is strategy-proof for every agent $m \in M$. 

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2.2 Obvious strategy-proofness

In this section, we briefly describe machinery recently developed in great generality by Li (2015), which we heavily rephrase for the special case of deterministic matching mechanisms with finite preference and outcomes sets. For ease of presentation, we furthermore focus on mechanisms under complete information. We emphasize that the results of this paper, and their proofs, still hold (mutatis mutandis) for the general definitions of Li (2015).\footnote{Indeed, readers who are familiar with the general definitions of Li (2015) may easily verify that if a randomized stable OSP mechanism exists, then derandomizing it by fixing in advance each choice of nature to some choice made with positive probability yields a deterministic stable OSP mechanism. Furthermore, if some stable mechanism is OSP under partial information, then it is also OSP under complete information.}

Definition 5 (Matching mechanism). A (single-sided extensive-form) matching mechanism for $M$ over $W$ consists of:

1. A rooted tree $T$.
2. A map $X : L(T) \rightarrow \mathcal{M}(M, W)$ from the leaves of $T$ to matchings between $M$ and $W$.
3. A map $Q : V(T) \setminus L(T) \rightarrow M$, from internal nodes of $T$ to $M$.
4. A map $A : E(T) \rightarrow 2^{\mathcal{P}(W)}$, from edges of $T$ to predicates over $\mathcal{P}(W)$, s.t. both of the following hold:
   - The predicates corresponding to edges outgoing from the same node are disjoint.
   - The disjunction (i.e., set union) of all predicates corresponding to edges outgoing from a node $n$ equals the predicate corresponding to the last edge outgoing from a node labeled $Q(n)$ along the path from the root to $n$, or to the predicate matching all elements of $\mathcal{P}(W)$ if no such edge exists.

Definition 6 (Pass through). We say that a preference profile $\bar{p} \in \mathcal{P}(W)^M$ passes through a node $n \in V(T)$ if for each edge $e$ along the path from the root to $n$, it is the case that $p_{Q(n')} \in A(e)$, where $n'$ is the source node of $e$.

Definition 7 (Implemented matching rule). Given an extensive-form matching mechanism $\mathcal{I}$, we denote by $C^\mathcal{I}$, called the matching rule implemented by $\mathcal{I}$, the (single-sided) matching rule mapping a preference profile $\bar{p} \in \mathcal{P}(W)^M$ to the matching $X(n)$, where $n$ is the unique leaf through which $\bar{p}$ passes. Equivalently, $n$ is the node in $T$ obtained by traversing $T$ from its root, and from each node $n'$ following the edge outgoing from $n'$ whose predicate matches the preference list of $Q(n')$.

Definition 8 (Divergence). We say that $p, p' \in \mathcal{P}(W)$ diverge at a node $n \in V(T)$ if there exist two distinct edges $e, e'$ outgoing from $n$ s.t. $p \in A(e)$ and $p' \in A(e')$.

Definition 9 (Obvious strategy-proofness (OSP)). Let $\mathcal{I}$ be an extensive-form matching mechanism.
1. $\mathcal{I}$ is said to be obviously strategy-proof (OSP) for an agent $m \in M$ if for every node $n$ with $Q(n) = m$ and for every $\bar{p} = (p_m)_{m \in M} \in \mathcal{P}(W)^M$ and $\bar{p}' = (p'_m)_{m \in M} \in \mathcal{P}(W)^M$ that both pass through $n$ s.t. $p_m$ and $p'_m$ diverge at $n$, it is the case that $C^I_m(\bar{p}) \succeq_m C^I_m(\bar{p}')$ according to $p_m$. In other words, the worst possible outcome for $m$ when acting truthfully (i.e., according to $p_m$) at $n$ is no worse than the best possible outcome for $m$ when misrepresenting his preference list at $n$ to be $p'_m$.

2. $\mathcal{I}$ is said to be obviously strategy-proof (OSP) if it is obviously strategy-proof for every agent $m \in M$.

Li (2015) shows that obviously strategy-proof mechanisms are, in a precise sense, mechanisms that can shown to implement strategy-proof matching rules under a cognitively limited proof model that does not allow for contingent reasoning.

Remark 1. To observe that strategy-proofness of $C^I$ for an agent $m \in M$ indeed is a weaker condition than obvious strategy-proofness of $\mathcal{I}$ for $m$, note that $C^I$ is strategy-proof for $m$ iff for every node $n$ with $Q(n) = m$ and for every $\bar{p} = (p_m)_{m \in M} \in \mathcal{P}(W)^M$ that passes through $n$ and for every $\bar{p}'_m \in \mathcal{P}(W)$ that diverges from $p_m$ at $n$,\(^8\) it is the case that $C^I_m(\bar{p}) \succeq_m C^I_m(\bar{p}')$ according to $p_m$.

Definition 10 (OSP-implementability). A (single-sided) matching rule $C : \mathcal{P}(W)^M \rightarrow \mathcal{M}(M, W)$ is said to be OSP-implementable if $C = C^I$ for some obviously strategy-proof matching mechanism $\mathcal{I}$. In this case, we say that $\mathcal{I}$ OSP-implements $C$.

2.3 The stable matching problem

Gale and Shapley (1962) consider a matching problem with two-sided preferences; they think of $M$ as a set of men, and of $W$ as a set of women. We consider a simplified version of their model.

Definition 11 (Preferences for $W$ over $M$). We define preference lists over $M$ and preference profiles for $W$ over $M$ analogously to Definition 1.

Definition 12 (Stability). Given a preference profile $\bar{p} = (p_m)_{m \in M} \in \mathcal{P}(W)^M$ for $M$ over $W$ and a preference profile $\bar{q} = (q_m)_{m \in M} \in \mathcal{P}(M)^W$ for $W$ over $M$, a matching $\mu \in \mathcal{M}(M, W)$ is said to be unstable w.r.t. $\bar{p}$ and $\bar{q}$ if there exist $m \in M$ and $w \in W$, each preferring the other over the partner matched to them by $\mu$, or if some participant $p \in M \cup W$ is matched to some other participant not on $p$’s preference list. If $\mu$ is not unstable, then it is said to be stable.

Theorem 1 (Gale and Shapley, 1962). A stable matching between $M$ and $W$ always exists for every pair of preference profiles $\bar{p} \in \mathcal{P}(W)^M$ and $\bar{q} \in \mathcal{P}(M)^W$. Moreover, there exists an $M$-optimal stable matching, i.e., a stable matching s.t. each man weakly prefers his match (or lack thereof) in this stable matching over his match (or lack thereof) in any other stable matching.

\(^8\)These conditions imply that $(p_m, \bar{p}_m)$ also passes through $n$. 
We now relate the concept of stability to the single-sided matching rules defined in Section 2.1.

**Definition 13** ($\bar{q}$-stability; $C^{\bar{q}}$). Let $\bar{q} \in \mathcal{P}(M)^W$ be a preference profile for $W$ over $M$.

- A (single-sided) matching rule $C$ is said to be $\bar{q}$-stable if for every preference profile $\bar{p} \in \mathcal{P}(W)^M$ for $M$ over $W$, the matching $C(\bar{p})$ is stable w.r.t. $\bar{p}$ and $\bar{q}$. A (single-sided) matching mechanism is said to be $\bar{q}$-stable if the matching rule that it implements is stable.

- We denote by $C^{\bar{q}} : \mathcal{P}(W)^M \rightarrow \mathcal{M}(M,W)$ the (single-sided, $\bar{q}$-stable) matching rule mapping each preference profile $\bar{p} \in \mathcal{P}(W)^M$ for $M$ over $W$ to the $M$-optimal stable matching w.r.t. $\bar{p}$ and $\bar{q}$.

Dubins and Freedman (1981) show that Gale and Shapley’s algorithm for finding the $M$-optimal stable matching is strategy-proof for all men (and in fact weakly group strategy-proof for the men). Gale and Sotomayor (1985) show that the $W$-optimal stable matching, unless it coincides with the $M$-optimal stable matching, is not strategy-proof for all men; their proof in fact shows that every stable matching rule that does not coincide with the $M$-optimal stable matching is not strategy-proof for all men (this result also follows from a much more general theorem of Chen et al., 2015). We now rephrase both of these results in the notation of this paper.

**Theorem 2** (Dubins and Freedman, 1981). For every preference profile $\bar{q} \in \mathcal{P}(M)^W$ for $W$ over $M$, the $M$-optimal stable matching rule $C^{\bar{q}}$ is strategy-proof.

**Theorem 3** (Gale and Sotomayor, 1985; Chen et al., 2015). For every preference profile $\bar{q} \in \mathcal{P}(M)^W$ for $W$ over $M$, no $\bar{q}$-stable matching rule $C \neq C^{\bar{q}}$ is strategy-proof.

In this paper, we focus on the question of whether $C^{\bar{q}}$ is not only strategy-proof, as Dubins and Freedman have shown, but also OSP-implementable. (As it is the unique the $\bar{q}$-stable matching rule, it is the only candidate for OSP-implementability.)

### 3 OSP-implementable special cases

Before stating our main impossibility result, we first review a few special cases in which $C^{\bar{q}}$, the $M$-optimal stable matching rule for fixed women’s preferences $\bar{q}$, is in fact OSP-implementable. For simplicity, we describe all of these cases under the assumption that the market is balanced (i.e., that $|W| = |M|$) and that all preference lists are full (i.e., that each participant prefers being matched to anyone over being unmatched); generalizing each of the below cases for unbalanced markets or for preference lists that are not full.

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9Indeed, a man who is not matched to his $M$-optimal partner under some given preferences for all participants can manipulate by truncating his preference list.
is straightforward. The first case we consider is that in which women’s preferences are perfectly aligned.

**Example 1** ($C^q$ is OSP-implementable when women’s preferences are perfectly aligned). Let $q \in P(M)$ and let $\bar{q} = (q)_{w \in W}$ be the preference profile in which all women share the same preference list $q$. $C^q$ is OSP-implementable by the following “serial-dictatorship” mechanism: ask the man most preferred according to $\bar{q}$ which woman he prefers most, and assign that woman to this man (in all leaves of the subtree corresponding to this response), ask the man second-most preferred according to $\bar{q}$ which woman he prefers most out of those not yet assigned to any man, and assign that woman to this man (in all leaves of the subtree corresponding to this response), etc. This mechanism can be shown to be OSP by the same reasoning that Li (2015) uses to show that *random serial dictatorship* (where the order of the dictatorship is determined uniformly at random rather than using an externally given order $q$) is OSP.

Another noteworthy example is that of arbitrary preferences in a very small matching market.

**Example 2** ($C^q$ is OSP-implementable when $|M| = |W| = 2$). When $|M| = |W| = 2$, $C^q$ is OSP-implementable for every $\bar{q} \in P(M)^W$. Indeed, let $M = \{a, b\}$ and $W = \{x, y\}$. If $q_x = q_y$, then $C^q$ is OSP-implementable as explained in Example 1. Otherwise, w.l.o.g. $a \succ_x b$ and $b \succ_y a$; for this case, Fig. 1 describes an OSP mechanism that implements $C^q$.

![Figure 1: An OSP mechanism that implements $C^q$ for $|W| = |M| = 2$ and for $\bar{q}$ where $a \succ_x b$ and $b \succ_y a$.](image)

The preference profiles in Examples 1 and 2 are special cases of the class of acyclical preference profiles, whose structure was defined by Ergin (2002).

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10Indeed, asking any man whether he prefers being unmatched over every (remaining not-yet-matched) woman never violates obviously strategy-proofness.
**Definition 14** (Acyclicity). A preference profile \( \bar{q} \in \mathcal{P}(M)^W \) for \( W \) over \( M \) is said to be **cyclical** if there exist \( a, b, c \in M \) and \( x, y \in W \) s.t. \( a \succ_x b \succ_x c \succ_y a \). If \( \bar{q} \) is not cyclical, then it is said to be **acyclical**.

Ergin (2002) shows that acyclicity of \( \bar{q} \) is necessary and sufficient for \( C^{\bar{q}} \) to be group strategy-proof (and not merely weakly group strategy-proof) and Pareto efficient. We now generalize Examples 1 and 2 by showing that acyclicity of \( \bar{q} \) (as in both of these examples) is sufficient for \( C^{\bar{q}} \) to also be OSP-implentable.

**Theorem 4** (Positive result for acyclical preferences). \( C^{\bar{q}} \) is OSP-implementable for every acyclical preference profile \( \bar{q} \in \mathcal{P}(M)^W \) for \( W \) over \( M \).

**Proof sketch.** We prove the result by induction over \( |M| = |W| \). By acyclicity, at most two men are ranked by some woman as her top choice. If only one such man \( m \in M \) exists, then he is ranked by all women as their top choice — in this case, similarly to Example 1, we ask this man for his top choice \( w \in W \), assign her to him, and then continue by induction (finding in an OSP manner the \( M \)-optimal stable matching between \( M \setminus \{m\} \) and \( W \setminus \{w\} \)). Otherwise, there are precisely two men \( a \in M \) and \( b \in M \) who are ranked by some woman as her top choice. By acyclicity, each woman either has \( a \) as her top choice and \( b \) as her second-best choice, or vice versa. We conclude somewhat similarly to Fig. 1: for each woman \( w \in W \) that prefers \( a \) most, we ask \( a \) whether he prefers \( w \) most; if so, we assign \( w \) to \( a \) and continue by induction. Otherwise, for each woman \( w \in W \) that prefers \( b \) most, ask \( b \) whether he prefers \( w \) most; if so, we assign \( w \) to \( b \) and continue by induction. Otherwise, we ask each of \( a \) and \( b \) for his top choice, assign each of them his top choice, and continue by induction. 

We conclude this section by noting, however, that acyclicity of \( \bar{q} \) is not a necessary condition for OSP-implementability of \( C^{\bar{q}} \), as demonstrated by the following example.

**Example 3** (OSP-implementable \( C^{\bar{q}} \) with cyclical \( \bar{q} \)). Let \( M = \{a, b, c\} \) and \( W = \{x, y, z\} \). We claim that \( C^{\bar{q}} \), for the following cyclical preference profile \( \bar{q} \), is OSP-implementable:

\[
\begin{align*}
    a & \succ_x b \succ_x c \\
    a & \succ_y c \succ_y b \\
    b & \succ_z a \succ_z c.
\end{align*}
\]

We begin by noting that \( \bar{q} \) is indeed cyclical, as \( a \succ_y c \succ_y b \succ_z a \). We now note that the following mechanism OSP-implements \( C^{\bar{q}} \):

1. Ask \( a \) whether he prefers \( x \) the most; if so, assign \( x \) to \( a \) and continue as in Example 2 (finding in an OSP manner the \( M \)-optimal stable matching between \( \{y, z\} \) and \( \{b, c\} \)).

2. Ask \( a \) whether he prefers \( y \) the most; if so, assign \( y \) to \( a \) and continue as in Example 2. (Otherwise, we deduce that 1) \( a \) prefers \( z \) the most and therefore 2) \( c \) will not end up being matched to \( z \).)

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3. Ask $b$ whether he prefers $z$ the most; if so, assign $z$ to $b$ and continue as in Example 2.

4. Ask $b$ whether he prefers $x$ the most; if so, assign $x$ to $b$, $z$ to $a$, and $y$ to $c$. (Otherwise, we deduce that $b$ prefers $y$ the most.)

5. Ask $c$ whether he prefers $x$ over $y$. If so, assign $x$ to $c$, $y$ to $b$, and $z$ to $a$. (Otherwise, we deduce that $b$ will not end up being matched to $y$.)

6. Ask $b$ whether he prefers $z$ over $x$. Assign $b$ to his most-preferred choice between $z$ and $x$ and continue as in Example 2.

Nonetheless, as we show in the next section, when there are more than 2 participants on each side and women’s preferences are sufficiently unaligned, $C^q$ is not OSP-implementable.

4 Impossibility result for general preferences

We now phrase our main impossibility result.

Theorem 5 (Impossibility result for general preferences).

a. If $|M| \geq 3$ and $|W| \geq 3$, then there exists a preference profile $\bar{q} \in \mathcal{P}(M)^W$, s.t. no $\bar{q}$-stable (single-sided) matching rule is OSP-implementable.

b. If $|M| \geq 3$ and $|W| \geq 3$, then as $|M| + |W|$ grows, we have for $\bar{q} \sim U(\mathcal{P}(M)^W)$ that:

- With high probability no $\bar{q}$-stable (single-sided) matching rule is OSP-implementable.
- For every three distinct men $a, b, c \in M$, as $|W|$ grows, with high probability no $\bar{q}$-stable (single-sided) matching mechanism is OSP for $a, b,$ and $c$.
- If $|M| \leq \text{poly}(|W|)$, then with high probability no $\bar{q}$-stable (single-sided) matching mechanism is OSP for more than two men.

We note that Theorem 5 applies to any $\bar{q}$-stable (one-sided) matching rule, and not only to the $M$-optimal stable matching rule $C^q$. Before proving Theorem 5, we first prove a special case of Theorem 5(a), which cleanly demonstrates the construction underlying our proof of all parts of Theorem 5.

Lemma 1. For $|M| = |W| = 3$, there exists a preference profile $\bar{q} \in \mathcal{P}(M)^W$ s.t. no $\bar{q}$-stable (single-sided) matching rule is OSP-implentable.

Proof. Let $M = \{a, b, c\}$ and $W = \{x, y, z\}$. Let $\bar{q}$ be the following preference profile (where each woman prefers being matched to any man over being unmatched):

\[
\begin{align*}
    a & \succ_x b \succ_x c \\
    b & \succ_y c \succ_y a \\
    c & \succ_z a \succ_z b.
\end{align*}
\]  

\[\text{\textsuperscript{11}}\text{This result also holds, with the same proof, if } \bar{q} \text{ is drawn uniformly at random from the set of all full preferences (i.e., where each woman prefers being matched to any man over being unmatched).}\]
Assume for contradiction that an OSP mechanism $\mathcal{I}$ that implements a $\bar{q}$-stable matching rule $C^{\bar{q}}$ exists. Therefore, $C^{\bar{q}}$ is strategy-proof, and so, by Theorem 3, $C^{\bar{q}} = C^q$. We define:

$$p_a^1 \triangleq z \succ y \succ x \quad p_b^1 \triangleq x \succ z \succ y \quad p_c^1 \triangleq y \succ x \succ z$$

and set $P_a \triangleq \{p_a^1, p_a^2\}$, $P_b \triangleq \{p_b^1, p_b^2\}$, and $P_c \triangleq \{p_c^1, p_c^2\}$.

Following a proof technique of Li (2015), we note that if we “prune” the tree of $\mathcal{I}$ by replacing, for each edge $e$, the predicate $A(e)$ with the conjunction (i.e., set intersection) of $A(e)$ with the predicate matching all elements of $P_{Q(n)}$, where $n$ is the source node of $e$, and by consequently deleting all edges $e$ for which $A(e) = \bot$, we obtain, in a precise sense, a mechanism that implements $C^q$ where the preference list of each man $m \in M$ is a priori restricted to be in $P_m$.$^{12}$ By a proposition of Li (2015), since the original mechanism $\mathcal{I}$ is OSP, so is the pruned mechanism as well.

Let $n$ be the earliest (i.e., closest to the root) node in the pruned tree that has more than one outgoing edge (such a node clearly exists, since $C^q = C^q$ is not constant over $P_a \times P_b \times P_c$). By symmetry of $\bar{q}$, $P_a, P_b, P_c$, w.l.o.g. $Q(n) = a$. By definition of pruning, it must be the case that $n$ has two outgoing edges, one labeled $p_a^1$, and the other — $p_a^2$. We claim that the mechanism of the pruned tree is in fact not OSP. Indeed, for $p_a = p_a^2$ (the “true preferences”), $p_b = p_b^2$, and $p_c = p_c^1$, we have that $C_a^q(\bar{p}) = C_a^q(\bar{p}) = x$, yet for $p_a' = p_a^1$ (a “possible manipulation”), $p_b' = p_b^1$, and $p_c' = p_c^2$, we have that $C_a^q(\bar{p}') = C_a^q(\bar{p}') = y$, even though $C_a^q(\bar{p}') = y \succ_a x = C_a^q(\bar{p})$ according to $p_a$ (by definition of $n$, both $\bar{p}$ and $\bar{p}'$ pass through $n$, and $p_a$ and $p_a'$ diverge at $n$), and so the mechanism of the pruned tree indeed is not OSP — a contradiction.

**Proof sketch of Theorem 5.** Part a follows from a reduction to Lemma 1. Indeed, let $a, b, c$ be three distinct men and let $x, y, z$ be three distinct women. Let $\bar{q} \in \mathcal{P}(W)^M$ be a preference profile s.t. the preferences of $x, y, z$ satisfy Eq. (1) w.r.t. $a, b, c$ (with arbitrary preferences over all other men), and with arbitrary preferences for all other women. Assume for contradiction that a $\bar{q}$-stable OSP mechanism $\mathcal{I}$ exists.

We prune (see the proof of Lemma 1 for an explanation of pruning) the tree of $\mathcal{I}$ s.t. the only possible preference lists for $a, b, c$ are those in which they prefer each of $x, y, z$, over all other women, and the only possible preference list for all other men is empty.$^{13}$ Let $\bar{q}'$ is the preference profile given in Lemma 1; the resulting (pruned) mechanism is a $\bar{q}'$-stable matching mechanism for $a, b, c$ over $x, y, z$, and so, by Lemma 1, it is not OSP; therefore, by the same proposition of Li (2015) that is used in Lemma 1, neither is $\mathcal{I}$.

All items in Part b follow from a similar argument. Indeed, our proof of Part a in fact shows that if $\bar{q}$ satisfies Eq. (1) w.r.t. three men $a, b, c$ and three women $x, y, z$, then no

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$^{12}$The interested reader is referred to Appendix A for the precise definition of a mechanism and OSP when the domain of preferences is restricted.

$^{13}$Alternatively, one could set for all other men arbitrary preference lists that do not contain $x, y, z$.

$^{14}$Formally, it is a matching mechanism for $W$ over $M$ w.r.t. the pruned preferences, but can be shown to always leave all participants but $a, b, c$ and $x, y, z$, unmatched, and so can be thought of as a matching mechanism for $a, b, c$ over $x, y, z$. 

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\( \bar{q} \)-stable matching mechanism is OSP for \( a, b, \) and \( c \). For the third part, for instance, we note that for a fixed triplet of distinct men \( a, b, c \in M \), the probability that Eq. (1) is not satisfied by \( \bar{q} \) w.r.t. \( a, b, c \) and any three women \( x, y, z \) decreases exponentially with \( |W| \), while the number of triplets of men increases polynomially with \( |M| \).

\[ \square \]

5 Two-sided mechanisms

The formal setting studied in this paper so far is that of single-sided matching mechanisms (where the preferences of the women are fixed and \( a \) priori announced), which is arguably more interesting theoretically, as it naturally allowed us to ask questions such as for which preference profile \( \bar{q} \) for women (or with which probability, for a random preference profile \( \bar{q} \) for women) is \( C^{\bar{q}} \) OSP-implementable. (Such questions are also of practical interest, as they are essentially equivalent to questions such as for which preference profiles for the schools, or with which probability, is the student-proposing stable matching OSP-implementable.) Nonetheless, as shown in this section, our analysis immediately yields strong results also for the setting of two-sided mechanisms, where the participants include not merely the men but also the women.

**Definition 15** (Two-sided matching rule; Stability).

- A **two-sided matching rule** is a function \( C : \mathcal{P}(W)^M \times \mathcal{P}(M)^W \rightarrow \mathcal{M}(M,W) \), from preference profiles for \( M \) over \( W \) and for \( W \) over \( M \) to matchings between \( M \) and \( W \).

- A two-sided matching rule \( C \) is said to be **stable** if \( C(\bar{p},\bar{q}) \) is stable w.r.t. \( \bar{p} \) and \( \bar{q} \), for every \( \bar{p} \in \mathcal{P}(W)^M \) and \( \bar{q} \in \mathcal{P}(M)^W \). A two-sided matching mechanism\(^{15}\) is said to be stable if the two-sided matching rule that it implements is stable.

To make our above statement, regarding the one-sided model naturally allowing more for interesting and detailed questions, more precise, we now show how results from each of the models can be used to imply results for the other.

**Lemma 2** (Relation between single-sided and two-sided OSP mechanisms). *For every \( M' \subseteq M \), there exists a stable two-sided matching mechanism that is OSP for \( M' \) if and only if for every \( \bar{q} \in \mathcal{P}(W)^M \) there exists a \( \bar{q} \)-stable one-sided matching mechanism that is OSP for \( M' \).*

**Proof sketch.** \( \Rightarrow \): Assume that there exists a stable two-sided matching mechanism \( I \) that is OSP for \( M' \), and let \( \bar{q} \in \mathcal{P}(W)^M \). We prune (see the proof of Lemma 1 for an explanation of pruning) the tree of \( I \) s.t. the women’s preference profile is fixed to be \( \bar{q} \). The resulting (pruned) mechanism is a **single-sided** matching mechanism that is \( \bar{q} \)-stable and (by the same proposition of Li (2015) that is used in Lemma 1) OSP for \( M' \), as required.

\( \Leftarrow \): Assume that for every \( \bar{q} \in \mathcal{P}(M)^W \) there exists a \( \bar{q} \)-stable one-sided matching mechanism \( I^{\bar{q}} \) that is OSP for \( M' \). We construct a stable two-sided matching mechanism \( I \) as

\(^{15}\)The interested reader is referred to Appendix B for the precise definition of a two-sided mechanism and when such a mechanism is said to be OSP.
follows: first ask all women, in some order, for all of their preferences; the leaves of the tree so far are thus in one-to-one correspondence with preference profiles $\tilde{q} \in \mathcal{P}(M)^W$ that pass through them. Next, at each “interim leaf” $n^{\tilde{q}}$ corresponding to a preference profile $\tilde{q} \in \mathcal{P}(M)^W$ (that passes through it), construct a subtree that is identical to the tree of $I^{\tilde{q}}$, with $n^{\tilde{q}}$ as its root. It is straightforward to verify that the fact that each $I^{\tilde{q}}$ is $\tilde{q}$-stable and OSP for $M'$ implies that $I$ is stable and OSP for $M'$.

Indeed, Lemma 2 allows us to immediately obtain results in the two-sided model from our results in the one-sided model.\textsuperscript{16} For instance, by Theorem 4 and Lemma 2 we have:

**Corollary 1** (Positive result for $|M| = 2$ for two-sided mechanisms). If $|M| = 2$, then the two-sided $M$-optimal stable matching rule (i.e., the two-sided matching rule mapping each pair of preference profiles to the corresponding $M$-optimal stable matching) is OSP-implementable (by first querying the women, and then, given their preferences, continuing as in Theorem 4).

More interestingly, by Theorem 5 and Lemma 2 we immediately obtain a strong impossibility result for two-sided matching mechanisms:

**Corollary 2** (Impossibility result for two-sided mechanisms). If $|M| \geq 3$ and $|W| \geq 3$, then no stable two-sided matching rule is OSP-implementable for $M$. Moreover, no stable two-sided matching mechanism is OSP for more than two men.

As with Theorem 5, we note that Corollary 2 applies to any stable two-sided matching rule, and not only to the $M$-optimal stable matching.

\section{Conclusion}

We show that no stable matching mechanism is obviously strategy-proof for all participants of one side of the market, unless the other side’s preference profile is very much aligned. This suggests that the strategic mistakes observed in practice (Hassidim et al., 2015; Rees-Jones, 2015) may not be avoided by describing the men-optimal stable matching via a different algorithm or description, and therefore highlights the importance of actual clearing houses gaining the unwavering trust of participating agents, so that participants both act accordingly when they are advised that no strategic possibilities exist, and trust that the mechanism will be run as stated after preferences are collected.

For the special case where women’s preferences are acyclical, we describe an OSP mechanism that implements the men-optimal stable matching. It is interesting to compare and contrast this mechanism with OSP mechanisms for auctions. In binary allocation problems, such as private-value auctions with unit demand, procurement auctions with unit supply, and binary public good problems, Li (2015) shows that in every OSP mechanism, each buyer chooses, roughly speaking, between a fixed option (i.e., quitting) and a “moving” option that

\textsuperscript{16}As we alluded above, the converse is not as easy, e.g., neither Theorem 4 nor Theorem 5(b) are immediate corollaries of results that are naturally stated for two-sided mechanisms/matching rules.
is getting worse over time (i.e., an increasing price). In contrast, in the OSP mechanism that we construct for the men-optimal stable matching with acyclical women’s preferences, each man \( m \) is either assigned his (current) top choice, or chooses between a fixed option (i.e., being unmatched) and a “moving” option that is getting better over time: choosing any of the women who prefer \( m \) the most among all remaining (not-yet-matched) men. An interesting direction for future research may be to see whether the idea of choosing between a fixed option and an improving moving option may be used to construct OSP mechanisms for additional classes of social choice rules.

The question of bridging our negative and positive results via an exact characterization of how aligned the preference profile of the other side need be to support an obviously strategy-proof implementation remains an interesting open problem.

**Open Problem 1.** Find a succinct characterization for the preference profiles \( \bar{q} \) for which \( C^q \) is OSP-implementable.

Interestingly, while deferred acceptance is (even weakly group) strategy-proof and has an ascending flavor similar to that of ascending unit-demand auctions or clock auctions (which are all obviously strategy-proof), deferred acceptance is in fact not obviously strategy-proof implementable. As discussed in the introduction, it seems that the fact that stability, in contrast with efficiency for one side or welfare maximization for one side, increases the difficulty of reasoning over stable mechanisms. In this context, it is worth noting a recent line of work (Segal, 2007; Gonczarowski et al., 2015) that highlights a similar message in terms of complexity, i.e., that the communication complexity of finding or even verifying a stable matching, even probabilistically and approximately, is significantly higher than the deterministic communication complexity of exactly solving problems that seem similar at first glance, yet whose objective notion can be thought of as one-sided welfare maximization, such as finding (or verifying) a maximal matching in a graph (cf. Dobzinski et al., 2014, Appendix A). Indeed, in more than one way, stability is not an “obvious” objective.

**References**


A Mechanisms with restricted domains

In this appendix, we explicitly provide the adaptation of the definitions of Section 2.2 for a restricted domain of preferences, as used in the proof of Lemma 1. The differences from the definitions of Section 2.2 are marked with an underscore. We emphasize that these definitions, like those of Section 2.2, are also a special case of the definitions of Li (2015).

For every $m \in M$, fix a subset $P_m \subseteq P(W)$. Furthermore, define $P \triangleq \times_{m \in M} P_m$.

Definition 16 (Matching mechanism). A (single-sided extensive-form) matching mechanism for $M$ over $W$ w.r.t. $P$ consists of:

1. A rooted tree $T$.
2. A map $X : L(T) \to M(M, W)$ from the leaves of $T$ to matchings between $M$ and $W$.
3. A map $Q : V(T) \setminus L(T) \to M$, from internal nodes of $T$ to $M$.
4. A map $A : E(T) \to 2^{P(W)}$, from edges of $T$ to predicates over $P(W)$, s.t. both of the following hold:
   - The predicates corresponding to edges outgoing from the same node are disjoint.
   - The disjunction (i.e., set union) of all predicates corresponding to edges outgoing from a node $n$ equals the predicate corresponding to the last edge outgoing from a node labeled $Q(n)$ along the path from the root to $n$, or to the predicate matching all elements of $P_{Q(n)}$ if no such edge exists.\(^{17}\)

Definition 17 (Pass through). We say that a preference profile $\bar{p} \in P$ passes through a node $n \in V(T)$ if for each edge $e$ along the path from the root to $n$, it is the case that $p_{Q(n')} \in A(e)$, where $n'$ is the source node of $e$.

Definition 18 (Implemented matching rule). Given an extensive-form matching mechanism $\mathcal{I}$ w.r.t. $P$, we denote by $C^I$, called the matching rule implemented by $\mathcal{I}$, the (single-sided) matching rule mapping a preference profile $\bar{p} \in P$ to the matching $X(n)$, where $n$ is the unique leaf through which $\bar{p}$ passes. Equivalently, $n$ is the node in $T$ obtained by traversing $T$ from its root, and from each node $n'$ following the edge outgoing from $n'$ whose predicate matches the preference list of $Q(n')$.

Definition 19 (Divergence). We say that $p, p' \in P(W)$ diverge at a node $n \in V(T)$ if there exist two distinct edges $e, e'$ outgoing from $n$ s.t. $p \in A(e)$ and $p' \in A(e')$.\(^{18}\)

Definition 20 (Obvious strategy-proofness (OSP)). Let $\mathcal{I}$ be an extensive-form matching mechanism w.r.t. $P$.

\(^{17}\)In particular, this implies that the predicates corresponding to edges outgoing from a node $n$ are predicates over $P_{Q(n)}$.

\(^{18}\)In particular, this implies that $p, p' \in P_{Q(n)}$. 

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1. $\mathcal{I}$ is said to be *obviously strategy-proof* (OSP) for an agent $m \in M$ if for every node $n$ with $Q(n) = m$ and for every $\bar{p} = (p_m)_{m \in M} \in \mathcal{P}$ and $\bar{p}' = (p'_m)_{m \in M} \in \mathcal{P}$ that both pass through $n$ s.t. $p_m$ and $p'_m$ diverge at $n$, it is the case that $C^I_m(\bar{p}) \succeq_m C^I_m(\bar{p}')$ according to $p_m$. In other words, the worst possible outcome for $m$ when acting truthfully (i.e., according to $p_m$) at $n$ is no worse than the best possible outcome for $m$ when misrepresenting his preference list to at $n$ be $p'_m$.

2. $\mathcal{I}$ is said to be *obviously strategy-proof* (OSP) if it is obviously strategy-proof for every agent $m \in M$.

### B Two-sided mechanisms

In this appendix, we explicitly provide the adaptation of the definitions of Section 2.2 for two-sided mechanisms, where the participants include not merely the men but also the women, as in Section 5. The differences from the definitions of Section 2.2 are marked with an underscore. We emphasize that these definitions, like those of Section 2.2, are also a special case of the definitions of Li (2015). Define $\mathcal{P} \triangleq \mathcal{P}(W)^M \times \mathcal{P}(M)^W$. For every two-sided preference profile $\bar{r} = (\bar{p}, \bar{q}) \in \mathcal{P}$, we write $r_m = p_m$ for every $m \in M$ and $r_w = q_w$ for every $w \in W$.

**Definition 21** (Two-sided matching mechanism). A two-sided (extensive-form) matching mechanism for $M$ and $W$ consists of:

1. A rooted tree $T$.
2. A map $X : L(T) \to \mathcal{M}(M,W)$ from the leaves of $T$ to matchings between $M$ and $W$.
3. A map $Q : V(T) \setminus L(T) \to M \cup W$, from internal nodes of $T$ to participants $M \cup W$.
4. A map $A : E(T) \to 2^{\mathcal{P}(W)} \cup 2^{\mathcal{P}(M)}$, from edges of $T$ to predicates over $\mathcal{P}(W)$ or over $\mathcal{P}(M)$, s.t. both of the following hold:
   - The predicates corresponding to edges outgoing from the same node are disjoint.
   - The disjunction (i.e., set union) of all predicates corresponding to edges outgoing from a node $n$ equals the predicate corresponding to the last edge outgoing from a node labeled $Q(n)$ along the path from the root to $n$, or, if no such edge exists, to the predicate matching all elements of $\mathcal{P}(W)$ if $Q(n) \in M$ and all elements of $\mathcal{P}(M)$ if $Q(n) \in W$.\(^1\)

**Definition 22** (Pass through). We say that a two-sided preference profile $\bar{r} \in \mathcal{P}$ passes through a node $n \in V(T)$ if for each edge $e$ along the path from the root to $n$, it is the case that $r_{Q(n')} \in A(e)$, where $n'$ is the source node of $e$.

\(^1\)In particular, this implies that the predicates corresponding to edges outgoing from a node $n$ are predicates over $\mathcal{P}(W)$ if $Q(n) \in M$ and over $\mathcal{P}(M)$ if $Q(n) \in W$. 

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Definition 23 (Implemented matching rule). Given a two-sided extensive-form matching mechanism \( \mathcal{I} \), we denote by \( C^\mathcal{I} \), called the two-sided matching rule implemented by \( \mathcal{I} \), the two-sided matching rule mapping a two-sided preference profile \( \bar{r} \in \mathcal{P} \) to the matching \( X(n) \), where \( n \) is the unique leaf through which \( \bar{r} \) passes. Equivalently, \( n \) is the node in \( T \) obtained by traversing \( T \) from its root, and from each node \( n' \) following the edge outgoing from \( n' \) whose predicate matches the preference list of \( Q(n') \).

Definition 24 (Divergence). We say that \( r, r' \in \mathcal{P}(W) \cup \mathcal{P}(M) \) diverge at a node \( n \in V(T) \) if there exist two distinct edges \( e, e' \) outgoing from \( n \) s.t. \( r \in A(e) \) and \( r' \in A(e') \).

Definition 25 (Obvious strategy-proofness (OSP)). Let \( \mathcal{I} \) be a two-sided extensive-form matching mechanism. \( \mathcal{I} \) is said to be obviously strategy-proof (OSP) for a participant \( a \in M \cup W \) if for every node \( n \) with \( Q(n) = a \), for every \( \bar{r}, \bar{r}' \in \mathcal{P} \) that both pass through \( n \) s.t. \( p_a \) and \( p'_a \) diverge at \( n \), it is the case that \( C^\mathcal{I}_a(\bar{r}) \succeq_a C^\mathcal{I}_a(\bar{r}') \) according to \( r_a \). In other words, the worst possible outcome for \( a \) when acting truthfully (i.e., according to \( r_a \)) at \( n \) is no worse than the best possible outcome for \( a \) when misrepresenting his or her preference list at \( n \) to be \( r'_a \).

Definition 26 (OSP-implementability). A two-sided matching rule \( C : \mathcal{P} \rightarrow \mathcal{M}(M, W) \) is said to be OSP-implementable for a set of participants \( A \subseteq M \cup W \) if \( C = C^\mathcal{I} \) for some two-sided matching mechanism \( \mathcal{I} \) that is OSP for (every participant in) \( A \).

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\( ^{20} \)In particular, this implies that \( r, r' \in \mathcal{P}(W) \) if \( Q(n) \in M \) and that \( r, r' \in \mathcal{P}(M) \) if \( Q(n) \in W \).